ON THE COMPUTATION OF ZEROS OF ENTIRE FUNCTIONS BY MEANS OF SERIES IN PROBLEMS OF MECHANICS

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V.A. BORODIN
(Moscow)

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Analytical expressions for zeros of algebraic polynomials and of entire transcendental functions find many applications in solving some problems of mechanics (such as stability of fluid flows, breakup of fluid currents e.a.). These expressions can be obtained by expanding the zeros of a given function into a power series in terms of its parameters.

A considerable amount of literature dealing with this problem exists [1 to 12], but in most instances either usable expressions are not derived, or they are derived only for some particular function. Lakhtin obtains in [1] expressions for zeros of algebraic polynomials in terms of hypergeometric functions. In more recent work of Belardinelli [2] an exhaustive review of relevant investigations is given together with author's own results which, unfortunately, again are not in a form which could easily be applied to functions of sufficiently general type.

In the present paper an attempt is made to obtain, by elementary means, expansions of zeros of entire functions (including algebraic polynomials of any degree) in a form suitable for practical application. Several examples from mechanics and numerical methods show practical application of the obtained series.

1. Let

$$f(z) = p_0 + p_1 u + p_2 u^2 + \cdots \qquad (1.1)$$

where $u = z - \xi$ is an entire function of z, expanded in a series at the point ξ lying near one of the zeros of $f(z)$. By the definition of entire function, series (1.1) converges for any finite z and ξ.

We know (Hurwitz [13]) that if $f_1(z), f_2(z), \ldots$ is a sequence of functions analytic in some region D bounded by a simple closed contour, if $f_n(z) \to f(z)$ uniformly in D and $f(z) \not\equiv 0$, then the point z_0 lying within D is a zero of $f(z)$ if and only if it is a limit point of a set of zeros of the function $f_n(z)$.

Therefore, if $f(z)$ is an entire function and the series (1.1) representing it converges at any finite point of a plane, then the sequences of zeros of partial sums of the expansion (1.1) converge to the zeros of $f(z)$.

We shall represent a zero of $f(z)$ by an expansion

$$u = a_0 + a_1 p_0 + a_2 p_0^2 + \cdots \qquad (1.2)$$

in powers of p_0. Let us introduce the notation

$$u^s = (a_0 + a_1 p_0 + a_2 p_0^2 + \ldots)^s = c_{s0} + c_{s1} p_0 + c_{s2} p_0^2 + \ldots \qquad (1.3)$$

Here [14]

$$c_{s0} = a_0^s, \quad c_{sm} = \frac{1}{m a_0} \sum_{i=1}^{m} (si + i - m)\, a_i c_{s\,m-i} \quad \text{for} \quad m \geqslant 1 \qquad (1.4)$$

It can easily be shown that

$$c_{sm} = \sum \frac{s(s-1)\ldots[s-(\alpha_i+\alpha_j+\ldots+\alpha_k)+1]}{\alpha_i!\,\alpha_j!\ldots\alpha_k!} a_0^{s-(\alpha_i+\alpha_j+\ldots+\alpha_k)} a_i^{\alpha_i} a_j^{\alpha_j}\ldots a_k^{\alpha_k} \qquad (1.5)$$

Summation is performed here over all possible partitions of m into equal or unequal natural components

$$i\alpha_i + j\alpha_j + \ldots + k\alpha_k = m \qquad (1.6)$$

Inserting (1.2) into (1.1) and taking (1.3) into account, we obtain

$$\sum_{k=0}^{\infty} p_k \sum_{i=0}^{\infty} c_{k,i} p_0^i = 0 \qquad (1.7)$$

Equating to zero the coefficients of consecutive powers of p_0 in (1.7), we obtain a system of equations for the coefficients a_k

$$\sum_{k=1}^{\infty} p_k a_0^k = 0, \quad 1 + \sum_{k=0}^{\infty} p_k c_{k1} = 0 \qquad (1.8)$$

$$\sum_{k=1}^{\infty} p_k c_{kt} = 0 \quad (t = 2, 3, \ldots) \qquad (1.9)$$

First Eq. of (1.8) has one zero root $a_0 = 0$. Let us find the coefficients of (1.2) corresponding to this root. When $a_0 = 0$, Formula (1.5) becomes

$$c_{t,k}^{\circ} = \sum \frac{t!}{\alpha_1!\,\alpha_2!\ldots\alpha_p!} a_1^{\alpha_1} a_2^{\alpha_2}\ldots a_p^{\alpha_p} \qquad (1.10)$$

$$1\cdot\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + p\alpha_p = k, \quad \alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_p = t$$

$$c_{0,k}^{\circ} = 0, \quad c_{t,k}^{\circ} = 0 \quad \text{when} \quad t > k \qquad (1.11)$$

Taking into account (1.10) for $a_0 = 0$ we obtain, in place of (1.8) and (1.9),

$$a_0 = 0, \qquad 1 + p_1 a_1 = 0 \qquad p_1 a_s + \sum_{k=2}^{s} p_k c_{k,s}^{\circ} = 0 \qquad (s = 2, 3, \ldots) \qquad (1.12)$$

defining the coefficients of (1.2).

Defining the coefficients a_k from (1.12) we obtain

$$a_0 = 0, \quad a_1 = -\frac{1}{p_1}, \quad a_k = \frac{1}{p_1^{2k-1}} \sum A_k p_r^{\alpha_r} p_s^{\alpha_s}\ldots p_q^{\alpha_q} \quad (k = 2, 3, \ldots) \qquad (1.13)$$

where

$$A_k = (-1)^{\alpha_r+1} \frac{[2(k-1)-\alpha_r]!}{\alpha_r!\,\alpha_s!\ldots k!} \quad \text{when } r = 1$$

$$A_k = -\frac{(2k-2)!}{\alpha_r!\,\alpha_s!\ldots\alpha_q!\,k!} \quad \text{when } r \neq 1 \qquad (1.14)$$

Summation in (1.13) is performed over all possible partitions of a natural number

$$2(k-1) = r\alpha_r + s\alpha_s + \ldots + q\alpha_q \qquad (1.15)$$

into equal or unequal natural components, under the condition that

$$\alpha_r + \alpha_s + \ldots + \alpha_q = k - 1 \qquad (1.16)$$

A series resembling (1.2) with coefficients (1.13), was obtained by Heegman [8] (quoted by Bazhenov in [5]) for a real root of an algebraic equation. Putting

$$\frac{p_k}{p_1} = q_k, \quad q_0 q_2 = \eta \qquad (1.17)$$

and inserting it into (1.2) we obtain, after some transformations,

$$u = -q_0 \sum_{t=0}^{\infty} \frac{(2t)!}{t!\,(t+1)!}\eta^t +$$

$$+\sum (-1)^{\alpha_1+1} q_0^{m+1} 1^{\alpha_1} q_s^{\alpha_s} \dots q_p^{\alpha_p} \sum_{t=0}^{\infty} \frac{(2t+2m-\alpha_1)!}{\alpha_s! \dots \alpha_p!\, t!\,(t+m+1)!}\eta^t \qquad (1.18)$$

where summation in the second term is performed over all partitions of consecutive, even natural numbers $2m$, beginning with the number 4, into m possible natural components except 2

$$2m = \alpha_1 + s\alpha_s + \dots + p\alpha_p, \quad \alpha_1 + \alpha_s + \dots + \alpha_p = m \qquad (1.19)$$

Let us denote the internal sums of (1.18) by

$$\sigma_{2k,\,n}(\eta) = \sum_{t=0}^{\infty} \frac{(2t+2k)!}{t!\,(t+n)!}\eta^t, \quad \sigma_{2k+1,\,n}(\eta) = \sum_{t=0}^{\infty} \frac{(2t+2k+1)!}{t!\,(t+n)!}\eta^t \qquad (1.20)$$

and use them to expand (1.18) into a usable formula, This formula is

$$u = -q_0\sigma_{01} + q_0^3 q_3 \sigma_{3,3} - q_0^4 q_4 \sigma_{44} - \frac{1}{2} q_0^5 q_3^2 \sigma_{65} + q_0^5 q_5 \sigma_{55} +$$

$$+ q_0^6 q_3 q_4 \sigma_{76} - q_0^6 q_6 \sigma_{66} + \frac{1}{6} q_0^7 q_3^3 \sigma_{97} - \dots \qquad (1.21)$$

2. Series (1.20 can be expressed in terms of hypergeometric functions. We can easily see that (2.1)

$$\sigma_{2k,\,n}(\eta) = \frac{(2k)!}{n!} F\left(k+1,\ k+\frac{1}{2};\ n+1;\ 4\eta\right) \quad \left(F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n\, n!} z^n\right)$$

$$\sigma_{2k+1,\,n}(\eta) = \frac{(2k+1)!}{n!} F\left(k+1,\ k+\frac{3}{2};\ n+1;\ 4\eta\right) \qquad (2.2)$$

where F is a Gauss' hypergeometric function. Hypergeometric series become truncated at any values of k and n which may be encountered in the internal sums of (1.18), and can then be expressed in terms of algebraic functions.

Using transformation formulas given in ([14], p. 1057), we obtain

$$\sigma_{2k,\,n}(\eta) = \frac{(2k)!}{n!}\left\{\frac{\Gamma(n+1)\,\Gamma(n-2k-{}^1/_2)}{\Gamma(n-k)\,\Gamma(n-k+{}^1/_2)\,(4\eta)^{k+1}} \times\right.$$

$$\times F\left(k+1,\ k-n+1;\ 2k-n+{}^3/_2;\ \frac{4\eta-1}{4\eta}\right) + \qquad (2.3)$$

$$\left. + \frac{\Gamma(n+1)\,\Gamma(2k-n+{}^1/_2)\,(1-4\eta)^{n-2k-{}^1/_2}}{\Gamma(k+1)\,\Gamma(k+{}^1/_2)\,(4\eta)^n} F\left(-k,\ -k+\frac{1}{2};\ n-2k+\frac{1}{2};\ 1-4\eta\right)\right\}$$

$$\sigma_{2k+1,\,n}(\eta) = \frac{(2k+1)!}{n!}\left\{\frac{\Gamma(n+1)\,\Gamma(n-2k-{}^3/_2)}{\Gamma(n-k)\,\Gamma(n-k-{}^1/_2)\,(4\eta)^{k+1}} \times\right. \qquad (2.4)$$

$$\times F\left(k+1,\ k-n+1;\ 2k-n+\frac{5}{2};\ \frac{4\eta-1}{4\eta}\right) +$$

$$\left. \frac{\Gamma(n+1)\,\Gamma(2k-n+{}^3/_2)\,(1-4\eta)^{n-2k-{}^3/_2}}{\Gamma(k+1)\,\Gamma(k+{}^3/_2)\,(4\eta)^n} F\left(-k,\ -k-\frac{1}{2};\ n-2k-\frac{1}{2};\ 1-4\eta\right)\right\}$$

where $\Gamma(x)$ is a gamma function and all hypergeometric functions are expressed in a finite form. Finite expressions for series $\sigma_{k,\,s}$ appearing in the first terms of the expansion (1.21) calculated by means of (2.3) and (2.4), are

$$\sigma_{01} = \frac{1}{2\eta} - \frac{(1-4\eta)^{1/2}}{2\eta}, \quad \sigma_{33} = \frac{\eta-1}{2\eta^3} + \frac{1-3\eta}{2\eta^3(1-4\eta)^{1/2}}$$

$$\sigma_{44} = \frac{2\eta - 1}{2\eta^4} + \frac{2\eta^2 - 4\eta + 1}{2\eta^4 (1 - 4\eta)^{1/2}}, \quad \sigma_{65} = \frac{2 - 3\eta}{\eta^5} + \frac{10\eta^3 - 30\eta^2 + 15\eta - 2}{\eta^5 (1 - 4\eta)^{3/2}}$$

$$\sigma_{55} = - \frac{\eta^2 - 3\eta + 1}{2\eta^5} + \frac{5\eta^2 - 5\eta + 1}{2\eta^5 (1 - 4\eta)^{1/2}}$$

$$\sigma_{.6} = \frac{3\eta^2 - 12\eta + 5}{2\eta^6} + \frac{70\eta^3 - 105\eta^2 + 42\eta - 5}{2\eta^6 (1 - 4\eta)^{3/2}}$$

$$\sigma_{66} = \frac{1 - 6\eta + 9\eta^2 - 2\eta^3}{2\eta^6 (1 - 4\eta)^{1/2}} - \frac{3\eta^2 - 4\eta + 1}{2\eta^6}$$

$$\sigma_{97} = 18 \left\{ - \frac{3\eta^2 - 3\eta + 1}{\eta^7} + \frac{210\eta^4 - 420\eta^3 + 252\eta^2 - 60\eta + 5}{6\eta^7 (1 - 4\eta)^{5/2}} \right\} \tag{2.5}$$

Since an algebraic polynomial is a particular case of an entire function, the above expansions can be used for determination of the roots of algebraic equations.

Limiting ourselves to the term of (1.1) containing u^m, we shall consider an m-th order Eq.

$$f(z) = \sum_{k=0}^{m} p_k u^k = 0 \tag{2.6}$$

On substitution

$$q_0 q_2 = \omega_2, \; -q_0^2 q_3 = \omega_3, \; \ldots, \; (-1)^k q_0^{k-1} q_k = \omega_k, \; \ldots, \; u = -q_0 \zeta \tag{2.7}$$

Eq. (2.6) becomes

$$f(\zeta) = 1 - \zeta + \omega_2 \zeta^2 + \omega_3 \zeta^3 + \ldots + \omega_m \zeta^m = 0 \tag{2.8}$$

It is now easy to obtain, from (1.18), the following expression for a root of (2.8) with the smallest modulus

$$\zeta = \sum_{t_m=0}^{\infty} \frac{\omega_m^{t_m}}{t_m!} \sum_{t_{m-1}=0}^{\infty} \frac{\omega_{m-1}^{t_{m-1}}}{t_{m-1}!} \cdots \sum_{t_3=0}^{\infty} \frac{\omega_3^{t_3}}{t_3!} \sum_{t_2=0}^{\infty} \frac{(2t_2 + 3t_3 + \ldots + mt_m)!}{t_2! \, [t_2 + 2t_3 + \ldots + (m-1) t_m + 1]!} \omega_2^{t_2} \tag{2.9}$$

In particular, for a 2-nd, 3-rd, 4-th and 5-th degree Eqs., we obtain

$$\zeta = \sigma_{01}(\omega_2) \quad \text{for} \quad m = 2, \qquad \zeta = \sum_{s=0}^{\infty} \frac{\omega_3^s}{s!} \sigma_{3s,\, 2s+1}(\omega_2) \quad \text{for} \quad m = 3$$

$$\zeta = \sum_{r=0}^{\infty} \frac{\omega_4^r}{r!} \sum_{s=0}^{\infty} \frac{\omega_3^s}{s!} \sigma_{3s+4r,\, 2s+3r+1}(\omega_2) \quad \text{for} \quad m = 4 \tag{2.10}$$

$$\zeta = \sum_{k=0}^{\infty} \frac{\omega_5^k}{k!} \sum_{r=0}^{\infty} \frac{\omega_4^r}{r!} \sum_{s=0}^{\infty} \frac{\omega_3^s}{s!} \sigma_{3s+4r+5k,\, 2s+3r+4k+1}(\omega_2) \quad \text{for} \quad m = 5$$

In practice, small number of terms in Formulas (2.9) and (2.10) can be used, provided that the root is expanded at a point which is sufficiently close to it. This makes it possible to utilize the expansion (1.21) and Expressions (2.5).

3. We shall now determine the radii of convergence for internal sums $\sigma_{k,s}$ in (1.18), (1.21), (2.9) and (2.10) when the values of all indices except the indices of internal sums, are fixed. We can write any of these sums as

$$\sigma_{k,s} = \sum_{t=0}^{\infty} \frac{(2t + k)!}{t! \, (t + s)!} \omega_2^t \tag{3.1}$$

where, by our assumption, k and s are constants. Radius of convergence of (3.1) can be found using a well known formula

$$\frac{1}{\rho}=\lim_{t\to\infty}\frac{(2t+2+k)!\,t!\,(t+s)!}{(t+1)!\,(t+1+s)!\,(2t+k)!}$$

This yields the value $\rho = 1/4$, hence the sums listed above converge, if (provided also that (1.17) and (2.7) are taken into account)

$$|\omega_2| = \left|\frac{p_0 p_2}{p_1^2}\right| < \frac{1}{4} \tag{3.2}$$

holds.

Let us find the region of convergence for the considered sum, on the ξ-plane. Using (1.1) we can write the inequality (3.2) as

$$\left|\frac{2f(\xi)f''(\xi)}{f'^2(\xi)}\right| < 1 \tag{3.3}$$

Now let us put

$$2f(\xi)f''(\xi)=\sum_{k=0}^{2(m-1)}A_k\xi^k,\qquad f'^2(\xi)=\sum_{k=0}^{2(m-1)}B_k\xi^k \tag{3.4}$$

where m is the degree of a polynomial $f(z)$.

Using (3.4) we can obtain from (3.3) an equation of regions of convergence of internal sums in expressions for roots of an algebraic Eq.

$$F(R,\theta)=\frac{d_0}{2}+\sum_{k=1}^{2(m-1)}d_k\cos k\theta=0\qquad\left(d_k=2\sum_{s=0}^{2(m-1)-k}D_{s,\,s+k}R^{2s+k}\right) \tag{3.5}$$

$$D_{ij}=\begin{vmatrix}B_i & A_i\\ A_j & B_j\end{vmatrix}$$

It can be assumed that the series (2.9) converges when the polynomial (1.1) is expanded at the point ξ lying within a region of convergence (3.5).

4. Expansion (2.9) of a root of an algebraic m-th degree equation obtained previously is of sufficiently general form(*), and can be used to obtain exact solutions of equations soluble in radicals or in terms of some special functions in cases when summation of the obtained series is feasible. We shall now consider some examples.

a) For $m = 2$

$$f(z)=z^2+pz+q=0 \tag{4.1}$$

By (2.10) and (2.5) we have

$$\zeta=\sigma_{0,1}(\omega_2)=\frac{1-\sqrt{1-4\eta}}{2\eta} \tag{4.2}$$

where

$$\zeta=-(p/q)\,z,\quad \eta=q/p^2,\quad \text{or}\quad z=-{}^1\!/_2\,p+\sqrt{{}^1\!/_4\,p^2-q} \tag{4.3}$$

b) For $m = 3$

$$f(z)=z^3+pz+q=0 \tag{4.4}$$

By (2.10) and (1.20) we have

$$\zeta=\sum_{s=0}^{\infty}\frac{\omega_3^{\,s}}{s!}\sum_{t=0}^{\infty}\frac{(2t+3s)!}{t!\,(t+2s+1)!}\,\omega_2^{\,t} \tag{4.5}$$

where

$$\zeta=-\frac{p}{q}\,z,\qquad \omega_2=0,\qquad \omega_3=-\frac{q^2}{p^3} \tag{4.6}$$

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*) Equation which does not include the unknown in the first power ($p_1 = 0$), should be transformed using a displacement transformation, before (2.9) can be applied.

Substitution of (4.6) into (4.5) yields

$$z = -\frac{q}{p}\sum_{s=0}^{\infty}\frac{(3s)}{s!\,(2s+1)!}\left(-\frac{q^2}{p^3}\right)^s \tag{4.7}$$

Since

$$\frac{(3s)!}{s!\,(2s+1)!} = \frac{(1/3)_s\,(2/3)_s}{(3/2)_s\,s!}\left(\frac{27}{4}\right)^s$$

therefore (4.7) yields

$$z = -\frac{q}{p}F\left(\frac{1}{3},\ \frac{2}{3};\ \frac{3}{2};\ -t^2\right) \qquad \left(t = \left(\frac{27q^2}{4p^3}\right)^{1/2}\right) \tag{4.8}$$

Using a transformation formula (given in [14], p. 1057, Formula 9.131.1) we obtain

$$z = -\frac{q}{p}(t^2+1)^{-1/3}F\left(\frac{1}{3},\ \frac{5}{6};\ \frac{3}{2};\ \frac{t^2}{t^2+1}\right). \tag{4.9}$$

Applying to it the summation formula (given in [14], p. 1054, Formula 9.121.4) we have for $n = 1/3$,

$$z = -\frac{3q}{2pt}\{[t+(t^2+1)^{1/2}]^{1/3} - [-t+(t^2+1)^{1/2}]^{1/3}\} \tag{4.10}$$

which on substitution of t from (4.8) into it, becomes the well known Cardan's formula

$$z = \left(-\frac{q}{2} + \left(\frac{q^2}{4}+\frac{p^3}{27}\right)^{1/2}\right)^{1/3} + \left(-\frac{q}{2} - \left(\frac{q^2}{4}+\frac{p^3}{27}\right)^{1/2}\right)^{1/3} \tag{4.11}$$

c) Expansion of a root of

$$y^n + xy - 1 = 0 \tag{4.12}$$

is of interest. Applying (2.9) we obtain

$$y = \frac{1}{x}\sum_{t=0}^{\infty}\frac{(nt)!}{t!\,[(n-1)t+1]!}\left(-\frac{1}{x^n}\right)^t \tag{4.13}$$

or

$$y = \frac{1}{x}\,{}_{n-1}F_{n-2}\left(\frac{1}{n},\ \frac{2}{n},\ldots,\ \frac{n-1}{n};\ \frac{2}{n-1},\ \frac{3}{n-1},\ldots\right.$$
$$\left.\ldots,\ \frac{n-2}{n-1},\ \frac{n}{n-1};\ -\frac{n^n}{(n-1)^{n-1}}\frac{1}{x^n}\right) \tag{4.14}$$

A necessary and sufficient condition for (4.13) or (4.14) to converge, is

$$|x| > \frac{n}{(n-1)^{(n-1)/n}} \tag{4.15}$$

We know that (4.12) has an expansion due to Mellin [2]

$$y = \frac{1}{n}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\Gamma((1+\alpha)/n)}{\Gamma(\alpha+1)\,\Gamma(1+[1-(n-1)\alpha]/n)}x^{\alpha} \tag{4.16}$$

its condition of convergence being

$$|x| < \frac{n}{(n-1)^{n-1/n}} \tag{4.17}$$

We see that (4.13) supplements the expansion obtained by Mellin and enables us to obtain a solution at any value of parameter x.

5. Several examples of application of obtained results to some problems of mechanics and numerical methods, follow.

a) In [15] a characteristic Eq.

$$H^3 + p_2H^2 + p_1H + p_0 = 0$$

$$p_2 = -\frac{n_1+Mn_2}{1+M},\quad p_1 = \frac{n_2-D}{1+M},\quad p_0 = \frac{n_2D}{1+M},\quad n_1 = m+A,\quad n_2 = m-A$$

$$A = {}^1/_2(1 - e^{-2m}), \quad M = \frac{\rho_3}{\rho_1}, \qquad D = \frac{Mm^3}{W}, \quad W = \frac{\rho_2 h V^2}{\sigma} \tag{5.1}$$

was obtained, where m is a dimensionless wave number, M is the ration of densities of the fluids and W is the Weber number (ρ_2 is the density, h is the thickness of the boundary layer, V is velocity and σ is surface tension coefficient). We shall try to determine the dependence of the oscillatory increment i.e. of the imaginary part of a complex root of (5.1), on the wave number for various values of the Weber number W.

In [15] a graphical solution was obtained. From (5.1), we have

$$z^3 + b_1 z + b_0 = 0, \qquad H = z - {}^1/_3 p_2 \tag{5.2}$$

and

$$b_1 = \frac{3p_1 - p_2^2}{3}, \qquad b_0 = \frac{2p_2^3 - 9p_1p_2 + 27p_0}{27} \tag{5.3}$$

Also, $\mathrm{Im}(H) = \mathrm{Im}(z)$. It can easily be shown that the second and third root of (5.2) can be expressed in terms of the first (real) root, by

$$z_{2,3} = -{}^1/_2 z_1 \pm {}^1/_2 i \sqrt{3z_1^2 + 4b_1} \tag{5.4}$$

$$z_i = {}^1/_2 \sqrt{3z_1^2 + 4b_1} \tag{5.5}$$

To determine z_1, we can apply one of the Formulas (4.13) or (4.16). Eqs. (5.2) can be transformed into (4.12) by means of a substitution

$$z = y\,(b_0)^{1/3} \tag{5.6}$$

Subsequent calculation confirmed the results obtained in [15]. We see that the analytical method of solution utilised by us makes it possible to achieve any predetermined degree of accuracy.

b) In [16] a problem of the breakup of a current of viscous fluid. Characteristic equation has the form

$$\alpha^2 + \frac{2\nu k^2}{I_0(ka)}\left[I_1'(ka) - \frac{2kl}{k^2+l^2}\frac{I_1(ka)}{I_1(la)} I_1'(la)\right]\alpha = \frac{\sigma k}{\rho a^2}(1 - k^2a^2)\frac{I_1(ka)}{I_0(ka)}\frac{l^2-k^2}{l^2+k^2}$$
$$(l^2 = k^2 + a/\nu) \tag{5.7}$$

where a denotes radius of the current, ka is a dimensionless wave number, ρ is the density of the fluid, σ is surface tension coefficient, ν is kinematic viscosity, and α is the complex frequency of oscillations while $I_0(ka)$ and $I_1(ka)$ are Bessel functions of an imaginary argument. Author of [16] states that the equation is too complicated to be solved by analytical methods and only considers the limiting cases.

Let us introduce dimensionless parameters

$$m = ka, \qquad z = \alpha\left(\frac{a^3\rho}{\sigma}\right)^{1/2}, \qquad A^2 = Lp = \frac{a\rho\sigma}{\eta^2} \tag{5.8}$$

into (5.7). Then,

$$la = \sqrt{m^2 + Az} \tag{5.9}$$

and Eq. (5.7) assumes the form

$$z^2 + \frac{2m^2 z}{AI_0(m)}\left[I_1'(m) - \frac{2m\sqrt{m^2+Az}}{2m^2+Az}\,\frac{I_1(m)}{I_1(\sqrt{m^2+Az})}\,I_1'(\sqrt{m^2+Az})\right] =$$
$$= m(1-m^2)\frac{Az}{2m^2+Az}\frac{I_1(m)}{I_0(m)} \tag{5.10}$$

Expanding Bessel functions and their derivatives into series in zeros of their arguments we obtain from (5.10), after some transformations,

$$\sum_{t=0}^{\infty} M_t \zeta^t = 0 \tag{5.11}$$

where

$$\zeta = m^2 + Az, \qquad M_0 = \frac{m^4 I_0(m) - 6m^3 I_1(m) - m(1 - m^2) I_1(m) A^2}{I_0(m)}$$

$$M_1 = \frac{m^2(16 - 3m^2) I_0(m) - 2m(8 + 7m^2) I_1(m) - m(1 - m^2) I_1(m) A^2}{8 I_0(m)}$$

$$M_t = \frac{[16(t-1)t(t+1) + 8t(t+1)m^2 + m^4] I_0(m) - [8t(t+1)m + 2(3+4t)m^3] I_1(m)}{2^{2t} t!(t+1)! I_0(m)} -$$

$$- \frac{m(1 - m^2) I_1(m) A^2}{2^{2t} t!(t+1)! I_0(m)} \qquad \text{for} \quad (t = 2, 3, \ldots)$$

Let us find the oscillation increment for the case of flow of a current of a 75% aqueous solution of glycerine. In this case we have $A^2 = Lp = 33.5$. Fig. 1 shows the result of our computation. Broken curve is obtained from

$$z = -\frac{m^2}{A} + \left(\frac{m^4}{A^2} + \frac{1}{2} m^2 (1 - m^2)\right)^{1/2} \tag{5.12}$$

which results from the substitution of dimensionless parameters (5.8) into a simplified equation given in [16]. Fig. 1 also shows the position of maximum increment for the case when the current is composed of ideal fluid (Rayleigh). We see that the curves differ from

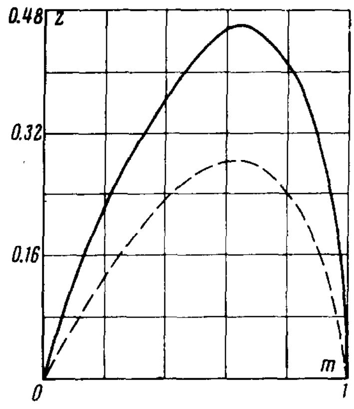

Fig. 1.

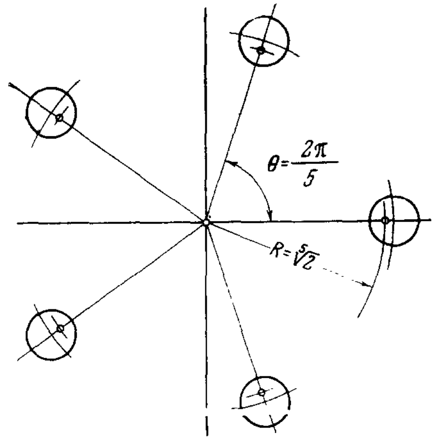

Fig. 2.

each other. Values of oscillation increments obtained by means of a more exact solution of characteristic equation are, roughly, one-and-half times as large as those obtained by means of (5.12) quoted in [16]. This, of course, gives much better idea of the length of the unbroken part of the considered current.

c) Consider the equation of dissection of a circle

$$f(z) = z^5 - a = 0 \tag{5.13}$$

Region of convergence of internal sums in (1.21) is, by (3.5), defined by Eq.

$$F(R, \theta) = -(64a^2 + 39R^{10}) + 128aR^5 \cos 5\theta = 0 \tag{5.14}$$

Fig. 2 shows regions of convergence of series for each of five roots of (5.13) for $a = 2$. Let us compute the real root of the equation, expanding $f(z)$ at the point $\xi = 1.2$ lying inside the region of convergence of the real root. Then, in place of (5.13), we obtain

$$\eta^5 + 6\eta^4 + 14.4\eta^3 + 17.28\eta^2 + 10.368\eta + 0.48832 = 0 \tag{5.15}$$

where

$$\eta = z - 1.2 \tag{5.16}$$

We shall use the first two terms of (1.21), on assumption that the remaining terms are small enough to be neglected. We obtain

$$q_0 = 0.047099, \quad q_2 = 1.66667, \quad q_3 = 1.38889,$$

$$w_2 = q_0 q_2 = 0.0784976, \quad \sigma_{0.1} = 1.093897, \quad \sigma_{3.3} = 1.56252$$

and finally

$$\eta = -0.05130 \text{ and } z = 1.14870 \tag{5.17}$$

with high degree of accuracy. This can easily be checked, as the real root of (5.13) for $a = 2$, is $z = 2^{0.2} = 1.148698$ (see e.g. [17]).

d) Let us find the smallest root of the equation [18]

$$f(z) = z^7 - \frac{7}{2} z^6 + \frac{63}{13} z^5 - \frac{175}{52} z^4 + \frac{175}{143} z^3 - \frac{63}{286} z^2 + \frac{7}{429} z - \frac{1}{3432} = 0 \tag{5.18}$$

Its roots can be located using any of the numerous existing methods. Graph of $f(z)$ based on approximate computations shown on Fig. 3, permits us to choose a point sufficiently near to the required root. Taking

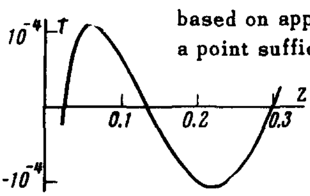

Fig. 3.

$$u = z - 0.02 \tag{5.19}$$

we obtain the coefficients of a transformed equation

$$q_0 = -0.00494670, \quad q_2 = -17.42303$$

$$q_3 = 109.73202, \quad \eta = q_0 q_2 = 0.0861865$$

Using again only the first two terms of (1.21) we obtain

$$u = 0.0054454 \quad \text{or} \quad z = 0.0254454 \tag{5.20}$$

while [18] gives its value as $z = 0.02544604$.

e) Let us find a real zero of the function

$$f(x) = \cos x \cosh x - 1 = 0 \tag{5.21}$$

Approximate values of zeros of $f(x)$ are given by [19]

$$a_n = {}^1/_2 (2n+1)\pi \tag{5.22}$$

where n denotes the n-th zero, when the trivial value $x = 0$ is disregarded.

Expansion of $f(x)$ at the point $x = 0$ has the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!} - 1 = 0 \tag{5.23}$$

which, on application of a formula ([14], p. 29, Formula 0.316) for multiplication of power series, becomes

$$f(\zeta) = \sum_{k=1}^{\infty} c_{2k} \zeta^k = 0 \quad \left(c_{2s} = 2 \sum_{t=0}^{s-1} \frac{(-1)^t}{(2t)!\,(4s-2t)!} + \frac{(-1)^s}{[(2s)!]^2} \right) \tag{5.24}$$

or ([14], p. 18, Formulas 0.153.1 and 0.153.3)

$$c_{2s} = (-1)^s \frac{2^{2s}}{(4s)!}, \quad \zeta = x^4 \tag{5.25}$$

Let us disregard the trivial solution $\zeta = 0$. To compute the zero x_n, we shall expand the function $f(\zeta)$ into a series

$$f(\zeta) = \sum_{k=0}^{\infty} c_{2(k+1)} \zeta^k = \sum_{k=0}^{\infty} p_k \eta^k, \quad \eta = \zeta - a_n^4 \tag{5.26}$$

obtaining

$$p_s = \frac{1}{s!} f^{(s)}(a_n^4) = \frac{1}{s!} \sum_{k=s}^{\infty} k(k-1)\ldots(k-s+1)\, c_{2(k+1)} a_n^{4(k-s)} \tag{5.27}$$

To obtain the first zero x_1, we shall again use first two terms of (1.21). The coefficients now will be

$$q_0 = -7.40712,\ q_2 = -4.2416\cdot 10^{-4},\ q_3 = 4.16592\cdot 10^{-8},$$
$$\eta = 3\cdot 1418\cdot 10^{-3},\ \sigma_{0,1} = 1.003162,\ \sigma_{3,3} = 1.0159,$$

and they yield the value of $\eta = x^4 - a_1{}^4 = 7.43052$.

Substituting into it the value of a_1 calculated according to (5.22), we obtain the first zero of (5.21) as $x_1 = 4.73004$, while [19] quotes $x_1 = 4.73$.

f) We shall find a real zero of the function

$$f(z) = J_0(2\sqrt{z}) = 0 \tag{5.28}$$

This problem was solved by Euler who expanded the zero into an infinite product; it is quoted in [20].

Let us use (3.3) to construct the regions of convergence for the expansion of (5.28). Fig. 4 shows the regions of convergence of series for the first three zeros of (5.28). A straight line parallel to the abscissa with the ordinate equal to ½, gives regions of convergence for three given zeros on the real axis. Function (5.28) can be expanded into a series at a zero, as follows

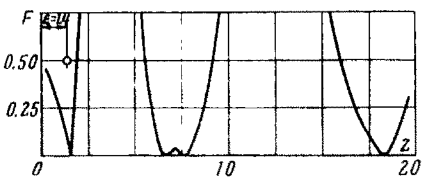

Fig. 4.

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(k!)^2} = 0 \tag{5.29}$$

Let us expand $f(z)$ at the point $\xi = 1.4$ lying within the region of convergence (Fig. 4). We obtain

$$q_0 = -0.045085370,\ q_2 = -0.34104094,\ q_3 = 0.043352828,$$
$$\eta = q_0 q_2 = 0.015375957,$$

which yields, for the first zero of (5.28), the value of $\alpha_1 = 1.4457964$. In [20] it is given as $\alpha_1 = 1.445796$.

BIBLIOGRAPHY

1. Lakhtin, L.K., Algebraic Equations Soluble in Terms of Hypergeometric Functions, Moscow, 1893.
2. Belardinelli, G., Fonctions hypergéométriques de plusiers variables et résolution analytique des équations algébriques générales. Paris, Gauthier – Villars, 1960.
3. Maksimovich., Expansion of Functions of Roots of an Equation into Series. Kazan', 1882
4. Chebyshev, P.L., Higher Algebra. M.-L., Izd. Akad. Nauk SSSR, 1936.
5. Bazenow, G.M., Über die Berechnung der Wurzeln von algebraischen Gléichungen mit Hilfe der unendlichen Reichen. Zap. Khar'k. mat. Tow. Ser. 4, Vol. 7, 1933.
6. Nemetti, V.P., A series for computing an incommeasurable root of an equation. Trudy. Mosk. Inst. inzh. transp. No. 10, pp. 7-11, 1929.
7. Shevelev, M.L., On the Whittaker method of determination of roots of an algebraic equation. Sb. "Matem. prosveshchenie" (Coll. Mathematical education). No. 12, pp. 35-47, 1937.
8. Heegmann, M.A., De la résolution générale des équations algébriques an moyen de séries. C. R. Acad. Sci., Paris, T. 52, pp. 972-974, 1861.
9. Kokareva, T.A., Approximate determination of the roots of equation $z^m = A + Bz^k$. Zh. vychisl. Mat. mat. Fiz., Vol. 2, No. 1, pp. 157-160, 1962.
10. Kodl Otakar., Řešeni rovnic potenčnimi řodami. Casop. pěstov. mat., 78, No. 3, s.263, 1953.
11. Markhashov, L.M., On the separation and computation of the roots of an algebraic

equation. PMM, Vol. 26, No. 4, 1962.
12. Mikeladze, Sh.E., Solution of Numerical Equations. Tbilisi, Metsniereba, 1965.
13. Titchmarsh, K., Theory of Functions. M.-L., Gostekhteoretizdat, 1951.
14. Gradshtein, I.S. and Ryzhik, I.M., Tables of Integrals, Sums, Series and Products. 4-th rev.ed. M., Fizmatgiz, 1962.
15. Borodin, V.A. and Iagodkin, V.I., Stability of motion of a plane boundary of separation of two fluids. PMTF, No. 1, 1967.
16. Levich, V.G., Physico-Chemical Hydrodynamics. M., Fizmatgiz, 1959.
17. Kompi, L.J., Chamber's six-figure mathematical tables. (Russian transl. M., "Nauka" 1964).
18. Krylov, A.N., Lectures of Approximate methods of Computation. 6-th ed. M., Gostekhizdat, 1954.
19. Maizel', V.M. ed., Mechanical Engineering Reference Handbook. Vol. 1, Mathematics, Khar'kov – Kiev, Gostekhizdat, 1937.
20. Watson, G.N., Theory of Bessel Functions, Part 1, M. (Russian translation) Izd. Inostr. lit., 1949.

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